

A QUASIFIBRATION OF SPACES OF POSITIVE SCALAR CURVATURE METRICS

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ABSTRACT. In this paper we show that for Riemannian manifolds with boundary the natural restriction map is a quasifibration between spaces of metrics of positive scalar curvature. We apply this result to study homotopy properties of spaces of such metrics on manifolds with boundary.

1. INTRODUCTION

The purpose of this note is to establish the fact that a natural restriction map in Riemannian geometry is a quasifibration of metrics of positive scalar curvature and discuss some applications to the study of spaces of positive scalar curvature metrics.

If M is an open manifold, then, by a result of Gromov [Gro69], there always exists on M a metric of positive sectional curvature. However, such a metric, in general, will not be complete. So, when studying metrics of positive scalar curvature on manifolds with boundary, it is necessary to impose some sort of boundary condition. It is natural to require that a metric is a product near the boundary.

Let M be a manifold with boundary ∂M . We fix a collar $c: \partial M \times (-1, 0] \rightarrow M$ and define the space $\mathcal{R}^+(M)$ of metrics of positive scalar curvature on M that restrict to a product metric near the boundary with respect to c . We take the usual Fréchet topology on this space. This topology is defined by the collection of C^k -norms $\|\cdot\|_k$ on the space of all Riemannian metrics $\mathcal{R}(M^n)$ with respect to some reference metric h : $\|g\|_k = \max_{i \leq k} \sup_{M^n} |\nabla^i g|$. The topology does not depend on the choice of the metric h .

By $\mathcal{R}_0^+(\partial M)$ we denote the image of $\mathcal{R}^+(M)$ under the restriction map

$$\rho: \mathcal{R}^+(M) \rightarrow \mathcal{R}^+(\partial M),$$

where $\rho(g) := g|_{\partial M}$. We assume that $\mathcal{R}^+(M)$ is non empty, which, of course, implies that $\mathcal{R}_0^+(\partial M)$ is non empty.

The definition of a quasifibration is due to Dold and Thom [DT58]. A surjective map $p: E \rightarrow B$ is a quasifibration if for all $x \in B$, all $y \in p^{-1}(x)$, and all $i \geq 0$ we have $\pi_i(E, p^{-1}(x), y) \approx \pi_i(B, x)$.

Theorem 1.1. *The map $\rho: \mathcal{R}^+(M) \rightarrow \mathcal{R}_0^+(\partial M)$ is a quasifibration.*

To show that a map between topological spaces $f: X \rightarrow Y$ (we take Y to be path connected) is a quasifibration it suffices to show that its fiber $f^{-1}(y_0)$ is homotopy equivalent to its homotopy fiber under the canonical inclusion map. The homotopy fiber of f is defined by replacing f by the Serre path fibration $\hat{f}: \hat{X} \rightarrow Y$ and taking the fiber $\Omega_{y_0} := \hat{f}^{-1}(y_0)$, some $y_0 \in Y$. Then $\Omega_{y_0} = \{(x, \omega)\}$ where a path

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$\omega: [0, 1] \rightarrow Y$ is such that $\omega(0) = f(x)$ and $\omega(1) = y_0$. Any two Ω_{y_0} and Ω_{y_1} are homotopy equivalent. It follows that the homotopy type of the homotopy fiber is well defined and we denote it as Ω .

The idea of the proof is to introduce an intermediate space Ω^s , which is defined by taking smooth paths ω in the definition of Ω . Then one can show that Ω^s is homotopy equivalent both to the fiber of ρ and to the homotopy fiber of ρ , see Lemmas 2.2 and 2.3.

One of important geometric implications of Theorem 1.1 is Theorem 1.2.

A Hausdorff space X is a topological manifold in the sense of Palais [Pal66] if there exists an open covering $\{O_\alpha\}$ of X and a family of maps $\{\theta_\alpha: O_\alpha \rightarrow V_\alpha\}$, where each V_α is locally convex topological vector space and each θ_α is a homeomorphism of O_α onto either an open subset of V_α or an open subset of a half space of V_α .

Theorem 1.2. *Let A be a contractible subset of $\mathcal{R}_0^+(\partial M)$. Suppose that A is a metrizable topological manifold. Then for any point $a \in A$ the inclusion $i: \rho^{-1}(a) \rightarrow \rho^{-1}(A)$ is a homotopy equivalence.*

Remark 1.1. The conclusion of the above Theorem is also true under the assumption that $\rho^{-1}(A)$ is an ANR or, more generally, is dominated by a CW-complex. The author does not know whether these properties follow from A merely being contractible.

In particular, if h_0 and h_1 are in the same path connected component of $\mathcal{R}_0^+(\partial M)$, then the spaces of positive scalar curvature metrics that near the boundary restrict to a product with correspondingly h_0 and h_1 are homotopy equivalent.

Another geometric consequence of Theorem 1.1 is an extension of results in [Che] to manifolds with boundary.

Let $N^{n-k} \subset M^n$, $k \geq 3$, be a submanifold of M^n . We assume that there exists a tubular neighborhood $\tau: N \times D^k \rightarrow M$, such that the restriction $\tau: \partial N \times D^k \rightarrow \partial M$ is a tubular neighborhood of ∂N in ∂M . We also assume that the collar c is compatible with N in the sense that its restriction to $\partial N \times (-1, 0]$ is a collar for ∂N .

We fix a metric g_N on N , and a torpedo metric g_0 on D^k (a torpedo metric in the disc D^k is an $O(k)$ -symmetric, positive scalar curvature metric, which is equal to a k -sphere metric near the center of the disc and is a product with a $(k-1)$ -sphere metric near the boundary of the disc), such that the metric $g_N + g_0$ has positive scalar curvature on $N \times D^k$. Here the fixed metric g_N can be any metric subject to the only requirement that it is a product near the boundary ∂N .

Let $h_0 \in \mathcal{R}_0^+(\partial M)$. Since codimension of ∂N in ∂M is greater than 2, from [Che] we may assume that $\tau^*(h_0) = g_N|_{\partial N} + g_0$. We define

$$(\rho^{-1}(h_0))_0 := \{g \in \rho^{-1}(h_0) \mid \tau^*(g) = g_N + g_0\}.$$

Theorem 1.3. *Suppose that $\mathcal{R}^+(M)$ is not empty. Then the inclusion map*

$$i: (\rho^{-1}(h_0))_0 \rightarrow \rho^{-1}(h_0)$$

is a homotopy equivalence.

Example 1.1. Let M^n be a manifold with a k handle $D^{n-k} \times D^k$, such that $k \geq 3$ and $\mathcal{R}^+(M^n)$ is nonempty. Let g_1 be a metric (which is a product near the boundary) on D^{n-k} and g_0 is a torpedo metric on D^k , such that $g := g_1 + g_0$

has positive scalar curvature. Then for any $h_0 \in \mathcal{R}_0^+(\partial M)$ the space $\rho^{-1}(h_0)$ is homotopy equivalent to the subspace of $\rho^{-1}(\hat{h}_0)$ consisting of metrics that restrict to the metric g on the handle. Here the metric \hat{h}_0 is obtained by deforming h_0 to be equal to $g_1|_{S^{n-k-1}} + g_0$ on $S^{n-k-1} \times D^k \subset \partial M$, see [Che] for details.

2. PROOFS OF THEOREMS

Given a smooth path of metrics $\alpha: I \rightarrow \mathcal{R}^+(X)$ on a closed smooth manifold X , we would like to put a positive scalar curvature metric on $X \times \mathbf{R}$. However, in general, the scalar curvature of the obvious metric $g(x, t) = \alpha(t)(x) + dt^2$ will not be positive.

We fix a smooth function $F: \mathbf{R} \rightarrow [0, 1]$ such that $0 \leq F' < 2$, $F(t) = 0$, $t \in (-\infty, \epsilon]$, $F(t) = 1$, $t \in [1 - \epsilon, \infty)$, for some $0 < \epsilon < 1/4$, and for a positive number τ we define a function $F_\tau: \mathbf{R} \rightarrow \mathbf{R}$, by $F_\tau(t) = F(\frac{t}{\tau})$.

Let

$$g_\tau^\alpha(x, t) := \alpha(F_\tau(t))(x) + dt^2.$$

Define

$$(1) \quad S'(\alpha) := \inf_{t>0} \{g_\tau^\alpha \text{ is a psc metric for all } \tau \geq t\},$$

$$S(\alpha) := \max(S'(\alpha), 1).$$

The function $S'(\alpha)$ is clearly upper semi-continuous. By the Lemma 2.1 below, it is also lower semi-continuous. It follows that S is continuous and defines a metric on $X \times \mathbf{R}$ by the formula

$$(2) \quad g_{t_0}^\alpha(x, t) := \alpha(F_{t_0}(t))(x) + dt^2,$$

where $t_0 := S(\alpha)$. By the same Lemma, this metric has positive scalar curvature and is a Riemannian product near $X \times 0$ and $X \times t_0$.

Lemma 2.1. *Let $\alpha: [0, 1] \rightarrow \mathcal{R}^+(X^n)$ be a C^∞ -family of positive scalar curvature metrics on a compact closed manifold X^n , then*

- (i) $\exists \lambda > 0$ such that $g^\lambda(t) := \alpha(F_\lambda(t)) + dt^2 \in \mathcal{R}^+(X^n \times [0, \lambda])$ and g^λ is a product metric near the boundary $(X \times 0) \cup (X \times 1)$ of $X \times [0, \lambda]$;
- (ii) if $t_0 = S(\alpha)$ is a positive number, then $\forall n = 1, 2, 3, \dots \exists t_n > 0$, $x_n \in X^n$, $\tau_n \in [0, t_n]$ such that, $t_0 - \frac{1}{n} < t_n < t_0$ and the scalar curvature of $\alpha(F_{t_n}(t)) + dt^2$ at (x_n, τ_n) is negative.

Proof. (i) Denote $g^\lambda(x, t) := \alpha(F_\lambda(t))(x) + dt^2$. Let (x_0, τ_0) be a point in $X^n \times [0, \lambda]$. Take normal coordinates for $\alpha(F_\lambda(\tau_0))$ at a point $x_0 \in X^n$. In these coordinates, we get $g_{ij}^\lambda(x_0, \tau_0) = \delta_{ij}$, $\Gamma_{ij}^k = 0$ for $1 \leq i, j, k \leq n$. Recall that

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Since $g_{i,n+1}^\lambda \equiv 0$ for $1 \leq i \leq n$, we get in our normal coordinates at (x_0, τ_0)

$$\begin{aligned} \Gamma_{ij}^k(x_0, \tau_0) &= 0 \quad \text{for } 1 \leq i, j, k \leq n \\ \Gamma_{n+1,i}^{n+1} &\equiv 0 \quad \text{for } 1 \leq i \leq n+1 \\ \Gamma_{n+1,j}^i(x_0, \tau_0) &= \frac{1}{2} \partial_{n+1} g_{ij}(x_0, \tau_0) \quad \text{for } 1 \leq i, j \leq n \\ \Gamma_{ij}^{n+1}(x_0, \tau_0) &= -\frac{1}{2} \partial_{n+1} g_{ij}(x_0, \tau_0) \quad \text{for } 1 \leq i, j \leq n. \end{aligned}$$

And the equation for sectional curvature are

$$\begin{aligned} R_{ijk}^s &= \partial_j \Gamma_{ik}^s - \partial_i \Gamma_{jk}^s + \Gamma_{ik}^l \Gamma_{jl}^s - \Gamma_{jk}^l \Gamma_{il}^s \\ R_{ijks} &= R_{ijk}^l g_{ls} \\ K_{ij} &= (\partial_i, \partial_j, \partial_i, \partial_j) = R_{ijij} \end{aligned}$$

From the Gauss equation for curvature we get for $1 \leq i, j \leq n$

$$K_{ij} = \bar{K}_{ij} + (b_{ii} b_{jj} - b_{ij}^2).$$

The remaining sectional curvatures

$$\begin{aligned} K_{i,n+1} &= R_{i,n+1,i}^{n+1} \\ &= \partial_{n+1} \Gamma_{ii}^{n+1} - \partial_i \Gamma_{n+1,i}^{n+1} + \Gamma_{ii}^l \Gamma_{n+1,l}^{n+1} - \Gamma_{n+1,i}^l \Gamma_{il}^{n+1} \\ &= \partial_{n+1} \Gamma_{ii}^{n+1} - \Gamma_{n+1,i}^l \Gamma_{il}^{n+1} \\ &= -\frac{1}{2} \partial_{n+1}^2 g_{ii} - \left(\frac{1}{2} \partial_{n+1} g_{ii}\right) \left(-\frac{1}{2} \partial_{n+1} g_{il}\right) \\ &= -\frac{1}{2} \partial_{n+1}^2 g_{ii} + \frac{1}{4} \sum_{l=1}^{n+1} (\partial_{n+1} g_{il})^2. \end{aligned}$$

Then the scalar curvature at a point (x_0, τ_0) is given by the formula

$$\begin{aligned} \kappa = \kappa_X &+ \sum_{i,j=1}^n (b_{ij}^2 - b_{ii} b_{jj}) \\ &- \sum_{i=1}^n \partial_{n+1}^2 g_{ii} + \frac{1}{2} \sum_{i,j=1}^n (\partial_{n+1} g_{ij})^2. \end{aligned}$$

Now, we have that $b_{ij} = \Gamma_{ij}^{n+1} = -\frac{1}{2} \partial_{n+1} g_{ij}(x_0, \tau_0)$ and the formulas for the derivatives

$$\begin{aligned} \partial_{n+1} g_{ij}(x_0, \tau_0) &= \frac{1}{t_0} F'(\tau_0) \alpha' \left(F \left(\frac{\tau_0}{t_0} \right) \right)_{ij} (x_0) \\ \partial_{n+1}^2 g_{ij}(x_0, \tau_0) &= \frac{1}{t_0^2} F''(\tau_0) \alpha' \left(F \left(\frac{\tau_0}{t_0} \right) \right)_{ij} (x_0) \\ &+ \frac{1}{t_0^2} (F'(\tau_0))^2 \alpha'' \left(F \left(\frac{\tau_0}{t_0} \right) \right)_{ij} (x_0) \end{aligned}$$

The scalar curvature for the product may now be expressed as

$$\begin{aligned} \kappa = \kappa_X &+ \frac{1}{4} \frac{1}{t_0^2} \sum_{i,j=1}^n \left(F' \left(\frac{\tau_0}{t_0} \right) \right)^2 ((\alpha'_{ij})^2 - \alpha'_{ii} \alpha'_{jj}) (x_0) \\ &+ \frac{1}{t_0^2} \sum_{i=1}^n \left(F'' \left(\frac{\tau_0}{t_0} \right) \alpha'_{ii}(x_0) + \left(F' \left(\frac{\tau_0}{t_0} \right) \right)^2 \alpha''_{ii}(x_0) \right) \\ &+ \frac{1}{2} \frac{1}{t_0^2} \sum_{i,j=1}^n \left(F' \left(\frac{\tau_0}{t_0} \right) \right)^2 (\alpha'_{ij}(x_0))^2 \end{aligned}$$

To finish the proof, notice that κ_X is positive for all $(x, \tau) \in X^n \times [0, t_0]$.

(ii) Suppose $t_0 > 0$ and let (x_0, τ_0) be a point in $X^n \times [0, t_0]$ where the scalar curvature is not positive. Such a point always exists since the $\kappa > 0$ is an open

condition and if the scalar curvature is everywhere positive we can find $t_1 < t_0$ such that the metric corresponding to t_1 will have positive scalar curvature. Now, freeze the values of t_0 , τ_0 , and x_0 which are in the arguments for the functions F , α and their derivatives, and regard the resulting function as a function of the inverse of t_0 . In the light of the argument above it has a positive derivative at t_0 , and its value at t_0 is less or equal than 0. So in an arbitrary neighborhood *on the left* from t_0 we can find a value t_n such that our function will be strictly negative at the point t_n . Now, “unfreezing” only τ_0 we can find a number τ_n such that $\frac{\tau_n}{t_n} = \frac{\tau_0}{t_0}$. The point (x_0, τ_n) is the one that we were seeking. \square

We fix a metric $h_0 \in \mathcal{R}_0^+(\partial M)$ and consider the homotopy fiber Ω of ρ at h_0 , $\Omega = \{(g, \omega) | \omega(0) = \rho(g), \omega(1) = h_0\}$. The topology on this fiber is the usual compact open topology. The smooth homotopy fiber Ω^s is defined analogously by taking ω to be a smooth path. We take the Fréchet topology on the smooth fiber.

There is a natural embedding i of the fiber $\rho^{-1}(h_0)$ into Ω^s , $i(g) = (g, *)$, where $*$ is the constant path $*(t) = h_0$.

Lemma 2.2. *The map $i: \rho^{-1}(h_0) \rightarrow \Omega^s$ is a homotopy equivalence.*

Proof. Let (g, ω) be a point in Ω^s and V_0 be a constant outward normal vector field on ∂M of the unit length. We take a smooth cutoff function ψ on \mathbf{R} with $\psi(-3/4) = 0$, $\psi(-1/4) = 1$, and define a vector field on M by setting

$$V(x) = \begin{cases} \psi(t)S(\omega)V_0 & x = c(a, t) \\ 0 & \text{otherwise} \end{cases},$$

where S is defined by the formula 1. Extend this vector field to $M \cup (\partial M \times [0, \infty))$ as a constant vector field $S(\omega)V_0$ on $\partial M \times [0, \infty)$ and denote by Φ_1^S the diffeomorphism determined by the flow of this vector field at $t = 1$. Then $\Phi_1^S(M) = M \cup (\partial M \times [0, S])$. Define

$$g^\omega = \begin{cases} g & \text{on } M \\ g_{t_0}^\omega & \text{on } \partial M \times [0, S] \\ h_0 & \text{on } \partial M \times [S, \infty) \end{cases},$$

where $g_{t_0}^\omega$ is given by the formula 2.

Now, we define an inverse map $r: \Omega^s \rightarrow \rho^{-1}(h_0)$ as

$$r(g, \omega) := (\Phi_1^{S(\omega)})^*(g^\omega).$$

Here we take the restriction of the pullback metric to M .

For $u \in [0, 1]$ we define a path $\omega_u(\tau) := \omega((1-u)\tau + u)$. Let g_{uF}^ω be the metric that is defined exactly as g^ω by taking the function uF instead of F . The homotopy $H: \Omega^s \times [0, 1] \rightarrow \Omega^s$ of $i \circ r$ to the identity map is given by

$$H((g, \omega), u) = \begin{cases} ((\Phi_1^{2uS(\omega)})^*(g_{0F}^\omega), \omega_0) & 0 \leq u \leq 1/2, \\ ((\Phi_1^{S(\omega)})^*(g_{(2u-1)F}^\omega), \omega_{2u-1}) & 1/2 \leq u \leq 1. \end{cases}$$

When u is equal to 0, the map $H(\cdot, 0)$ is the identity map on Ω^s . When $u = 1$, the map $H(\cdot, 1)$ is equal to $i \circ r$. \square

Lemma 2.3. *The inclusion map $i: \Omega^s \rightarrow \Omega$ is a homotopy equivalence.*

Proof. The proof is completely analogous to the proof of Theorem 17.1 in [Mil63]. Since Ω is an open subset of a locally convex topological vector space, we can cover it with convex open sets. Then take Ω_k , the space of all paths ω such that $\omega([(j-1)/2^k, j/2^k])$ is contained in some element of the covering. The space Ω is a homotopy direct limit of Ω_k and the space Ω^s is a homotopy direct limit of $\Omega_k^s := i^{-1}(\Omega_k)$. By Milnor's argument, the map

$$i|_{\Omega_k^s} : \Omega_k^s \rightarrow \Omega_k$$

is a homotopy equivalence. Here, the inverse map is defined by taking a path $\omega \in \Omega_k$ and assigning to it a piece-wise linear path that coincides with ω at points $j/2^k$. Then we smooth the resulting path by pre-composing with a smooth function that maps $j/2^k$ to $j/2^k$ and all of whose derivatives vanish at points $j/2^k$. This finishes the proof. \square

Proof of Theorem 1.2. If $p: E \rightarrow B$ is a quasifibration over a contractible space B then for any point $b \in B$ the inclusion of the fiber $p^{-1}(b) \rightarrow E$ induces a weak homotopy equivalence. From Palais [Pal66] we know that $\rho^{-1}(a)$ and $\rho^{-1}(A)$ are both dominated by CW-complexes. For such dominated spaces a weak homotopy equivalence is, in fact, a homotopy equivalence by a theorem of J. H. C. Whitehead. From Theorem 1.1 it follows that the inclusion map i is a homotopy equivalence. \square

Proof of Theorem 1.3. As in the proof of Theorem 1.2, it suffices to show that i is a weak homotopy equivalence. In [Che] a method for deforming compact families of metrics of positive scalar curvature was developed, which allowed to prove the weak homotopy equivalence in the case of closed manifolds M and N . This deformation can be readily adapted to manifolds with boundary and has an important property. Namely, it preserves the product structure with respect to the fixed tubular map τ , i.e. if $\tau^*(g) = g_N + g_0$, then for the deformation metrics $g(t)$, $t \in [0, 1]$ we have $\tau^*(g(t)) = g_N + g_0(t)$ and $g(t)$ is constant outside of the tubular neighborhood of N . The problem is that, in general, $g_0(t)$ is not equal to g_0 , so this deformation takes us outside the fiber $\rho^{-1}(h_0)$.

The solution is to introduce a subspace $A \subset \mathcal{R}_0^+(\partial M)$ consisting of metrics that are equal to h_0 outside $\tau(\partial N \times D^k)$ and equal to $g_N|_{\partial N} + g_w$ on $\tau(\partial N \times D^k)$. Here, g_w is a warped metric in the disc, i.e. $g_w = g(t)^2 dt^2 + f(t)^2 d\xi^2$, where $d\xi^2$ is the standard metric of the $(k-1)$ -sphere of radius 1, g is a smooth even function, and f is a smooth odd function. Note that $g_0 \in A$, i.e. a torpedo metric is a warped metric. Then, from the construction of the deformation, we have that $g_w(t)$ is a warped metric for all $t \in [0, 1]$. This allows us to conclude a weak homotopy equivalence (and, therefore, a homotopy equivalence) between $(\rho^{-1}(h_0))_0$ and $\rho^{-1}(A)$.

From [Che] it follows that the inclusion map $h_0 \rightarrow A$ is a weak deformation retraction, cf. Theorem 4.1 in [Che]. Now, the proof follows from Theorem 1.2. \square

REFERENCES

- [Che] Vladislav Chernysh, *On the homotopy type of the space $\mathcal{R}^+(M)$* , Preprint, arXiv: math.GT/0405235.
- [DT58] Albrecht Dold and René Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. (2) **67** (1958), 239–281.
- [Gro69] M. L. Gromov, *Stable mappings of foliations into manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 707–734.
- [Mil63] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963.

[Pal66] Richard S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16.

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